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LETTER TO THE EDITOR

Warped products and Einstein metrics

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Abstract
Warped product construction is an important method to produce a new metric with a base manifold and a fibre. We construct compact base manifolds with a positive scalar curvature which do not admit any non-trivial Einstein warped product, and noncompact complete base manifolds which do not admit any non-trivial Ricci-flat Einstein warped product.

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1. Introduction

Warped product is an important method to produce a new metric with a base manifold and a fibre. This method has been used for the construction of Einstein metrics on noncompact complete manifolds and other important examples in relativity and differential geometry [1, 3]. Construction of a non-trivial Einstein warped product on a compact manifold was questioned by Besse [2, section 9.103], but no examples were found yet. Recently, Mustafa proved that there exists a metric on every compact manifold N such that non-trivial Einstein warped products, with base N, cannot be constructed [8]. There is a known obstruction to the existence of a compact Einstein warped product. If a compact Einstein warped product manifold has a non-positive scalar curvature, then the warped product should be trivial [6]. In this letter, we construct compact base manifolds with a positive scalar curvature which do not admit any non-trivial Einstein warped product. For a noncompact complete Riemannian manifolds, we prove that if a base manifold has at most quadratic volume growth and non-positive total scalar curvature, then non-trivial Ricci-flat Einstein warped product cannot exist. The Riemannian–Schwarzschild metric is a Ricci-flat Einstein warped product with the base manifold (R^2, g), where (R^2, g) has a positive scalar curvature. Our result implies that the sign of scalar curvature of a noncompact base manifold plays an important role in the Einstein warped product.

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2. Warped products

Let $N = (N^n, g_N)$ and $F = (F^d, g_F)$ be two Riemannian manifolds and f be a positive smooth function on N. We denote by π and σ the projections of $N \times F$ onto N and F, respectively. The warped product $M = N \times_f F$ is the product manifold $M = N \times F$ furnished with the metric g defined by $g = \pi^* g_N + f^2 \sigma^* g_F$, where $*$ denotes the pull back. N is called the base of $M = N \times_f F$, and F the fibre, and the warped product is called trivial if f is a constant function. We denote by $\mathrm{Ric}^N, \mathrm{Ric}^F$ and H^f the lifts to M of the Ricci curvatures of N and F, and the Hessian of f, respectively. Throughout this letter, we let dim $N \geqslant 2$ and $d = \dim F \geqslant 2$. The warped product $(M, g) = N \times_f F$ is Einstein with $\mathrm{Ric} = \lambda g$ if and only if (F, g_F) is Einstein, i.e., $\mathrm{Ric}_F = \mu g_F$ for a constant μ and the followings hold [9, p 211]:

$$\lambda g_N = \mathrm{Ric}^N - \frac{d}{f} H^f, \tag{1}$$

$$\lambda = \frac{\mu}{f^2} - \frac{\Delta f}{f} - (d-1)\left|\frac{\nabla f}{f}\right|^2_{g_N}, \tag{2}$$

where $\Delta f = \mathrm{tr}\, H^f$.

First, we prove that there are infinitely many metrics on a base manifold N of dimension $m \geqslant 3$ such that $M = N \times_f F$ cannot be a non-trivial Einstein warped product. This comes from the resolution of the Yamabe problem on compact manifolds [10]:

Theorem 1 *(Schoen)***.** *There exists a conformal metric $\overline{g} = e^u g$ for any given metric (N, g) on a compact Riemannian manifold N of dimension $m \geqslant 3$ such that $\overline{g}$ has a constant scalar curvature, where u is a smooth function on N.*

Assume that scalar curvature S_N of N is constant. By taking the trace of (1),

$$m\lambda = S_N - d\frac{\Delta f}{f}. \tag{3}$$

Since f is positive, N is compact and there is no nonconstant super harmonic or subharmonic function on a compact manifold, f must be a constant on N.

Theorem 2. *For a given metric g_N on N, there exists a conformal metric $\overline{g} = \mathrm{e}^{2u} g$ such that there are no non-trivial Einstein warped products $N \times_f F$ with base $(N, \overline{g})$.*

A differential manifold of dimension $m \geqslant 3$ admits infinitely many different conformal classes. Therefore, there are infinitely many metrics on a base manifold N of dimension $m \geqslant 3$ such that there are no non-trivial Einstein warped products $N \times_f F$ with base N.

Next we construct a manifold N with a positive scalar curvature such that there are no non-trivial Einstein warped products $(M, g) = N \times_f F$ with base N. If a warped product $(M, g) = N \times_f F$ is Einstein with $\mathrm{Ric} = \lambda g$ with $\lambda \leqslant 0$, then the warped product should be trivial [6]. Therefore we consider only when $\lambda > 0$.

Theorem 3. *Let $\left(N_1^r, g_1\right)$ and $\left(N_2^m, g_2\right)$ be compact Riemannian manifolds with scalar curvature S_1 and S_2, respectively. Assume that S_1 is a positive constant and $\left(N_2^m, g_2\right)$ has a non-positive total scalar curvature, i.e., $\int_{N_2} S_2 \,\mathrm{d}V_{g_2} \leqslant 0$, with $|S_2| < S_1$. Consider $N = N_1 \times N_2$ with the product metric $g_N = g_1 + g_2$ and positive scalar curvature S_N. Then there are no non-trivial Einstein warped products $(M, g) = N \times_f F$ with base (N, g_N).*

Proof. Let $f = f(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+m})$ be a positive smooth function on $N_1^r \times N_2^m$. By taking the trace of (1) for $i = 1$ to $i = r$, and $i = r + 1$ to $i = r + m$, respectively, we have

$$r\lambda = S_1 - d\frac{\triangle_{g_1} f}{f}, \tag{4}$$

$$m\lambda = S_2 - d\frac{\triangle_{g_2} f}{f}. \tag{5}$$

Since S_1 is constant $\triangle_{g_1} f$ should have a fixed sign in (4), which implies that f does not depend on $x_1, \ldots, x_r$. Integrating (5) on N_2,

$$\begin{aligned}\int_{N_2} S_2 \, \mathrm{d}V_{g_2} &= \int_{N_2} m\lambda + d\frac{\triangle_{g_2} f}{f} \, \mathrm{d}V_{g_2} \\ &= \int_{N_2} m\lambda + d\left|\frac{\nabla f}{f}\right|^2_{g_2} \mathrm{d}V_{g_2},\end{aligned} \tag{6}$$

which is not possible since $\lambda > 0$ and $\int_{N_2} S_m \, \mathrm{d}V_{g_2} \leqslant 0$. We conclude that f is a constant function. □

Theorem 4. *Let (N_1^r, g_1) be a compact manifold of dimension $r \geqslant 3$ with constant scalar curvature S_1 and (N_k, g_k) be two-dimensional compact Riemannian manifolds for $k = 2, \ldots, m$, and denoted by $(N, g_N) = \left(\prod_{k=1}^{k=m} N_k, \sum_{k=1}^{m} g_k\right)$. If $N \times_f F$ is Einstein, then the warped product should be trivial.*

Proof. By taking the trace of (1) on $N_1, N_2, \ldots, N_m$ respectively,

$$r\lambda = S_1 - d\frac{\triangle_{g_1} f}{f}, \tag{7}$$

$$2\lambda = S_k - d\frac{\triangle_{g_k} f}{f} \qquad \text{for } k = 2, \cdots, m. \tag{8}$$

It is known that there is no non-trivial Einstein warped product over a compact two-dimensional base manifold, which can be found in [2, section 9.119] and [5]. Therefore, there is no nonconstant function on each N_k satisfying (8) and (2) for $k = 2, \ldots, m$, which implies that f is a function of N_1 only. Since f satisfies (7) and S_1 is a constant, f should be a constant by the argument in the above of theorem 2. □

Next we construct a noncompact complete base manifold (N, g_N) such that there is no non-trivial Ricci-flat Einstein warped product $M = N \times_f F$ with base (N, g_N). For this, we introduce some notations. Let (N, g_N) be a noncompact complete Riemannian manifold. For $\Omega \subset N$, we let $|\Omega|$ be the volume of Ω with respect to the metric g_N and $B(R) \equiv \{x \in N |\, \mathrm{dist}(p, x) \leqslant R\}$ for a fixed point $p \in N$. (N, g_N) has at most quadratic volume growth if there exists a constant c such that $\limsup_{R\to\infty} \frac{|B(R)|}{R^2} \leqslant c$. For the construction of (N, g_N), we estimate the solutions of (3) using the methods developed in [7].

Theorem 5. *Let (N, g_N) be a noncompact complete Riemannian manifold with scalar curvature S_N. Assume that (N, g_N) satisfies the following conditions:*

(a) (N, g_N) has at most quadratic volume growth,
(b) $-\infty \leqslant \int_N S_N \, \mathrm{d}V_{g_N} \leqslant 0$ and $\int_N S_N^+ \, \mathrm{d}V_{g_N}$ is finite, where $S_N^+(x) = \max(0, S_N(x))$. Then the followings hold:

(c) If $(N \times F, g_N + f^2 g_F)$ is Ricci-flat Einstein, then f should be a constant.
(d) If there exists a point q in N such that $S_N(q) < 0$, then $(N \times F, g_N + f^2 g_F)$ cannot be Ricci-flat Einstein.

Proof. Choose a smooth function ϕ^2 such that $0 \leqslant \phi \leqslant 1$ on N, $\phi = 1$ on $B(R)$, $\phi = 0$ outside of $B(2R)$ and $|\nabla\phi| \leqslant c/R$, where c is a constant. From the above condition (b), for any given small ϵ there exists a sufficiently large R_0 such that $\int_{N-B(R)} S_N^+ \,\mathrm{d}V_g \leqslant d\epsilon$ and $\int_{B(R)} S_N \,\mathrm{d}V_g < d\epsilon$ for $R \geqslant R_0$. This implies

$$\int_{B(2R)} S_N\phi^2 \,\mathrm{d}V_{g_N} \leqslant \int_{B(R)} S_N \,\mathrm{d}V_{g_N} + \int_{B(2R)-B(R)} S_N^+ \,\mathrm{d}V_{g_N}$$
$$< 2d\epsilon. \tag{9}$$

Multiplying ϕ^2 on (3) with $\lambda = 0$,

$$\int_N \phi^2 \left|\frac{\nabla f}{f}\right|^2_{g_N} - \frac{\phi^2}{d} S_N \,\mathrm{d}V_{g_N} = \int_N 2\phi\nabla\phi \cdot \frac{\nabla f}{f} \,\mathrm{d}V_{g_N} \tag{10}$$

$$\leqslant \int_N \frac{1}{2}\left|\frac{\nabla f}{f}\right|^2_{g_N} + 2|\nabla\phi|^2 \,\mathrm{d}V_{g_N}, \tag{11}$$

where the Hölder inequality is used in (11). Therefore,

$$\int_N \frac{\phi^2}{2}\left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N} \leqslant \int_N 2|\nabla\phi|^2 + \frac{\phi^2}{d} S_N \,\mathrm{d}V_{g_N} \tag{12}$$

$$\leqslant \frac{2c^2}{R^2}|B(2R) - B(R)| + 2\epsilon$$
$$\leqslant c', \tag{13}$$

where c' is some positive constant and the quadratic volume growth condition is used in (13). Therefore, the integral $\int_N \left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N}$ is uniformly bounded. From (10),

$$\int_N \frac{\phi^2}{2}\left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N} \leqslant \frac{2c}{R}\int_{B(2R)-B(R)} \left|\frac{\nabla f}{f}\right|_{g_N} \mathrm{d}V_{g_N} + \int_N \frac{\phi^2}{d} S_N \,\mathrm{d}V_{g_N}$$
$$\leqslant 2c\left(\int_{B(2R)-B(R)} \left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N}\right)^{1/2} \frac{|B(2R) - B(R)|^{1/2}}{R} + 2\epsilon. \tag{14}$$

By the condition (a) and $\int_N \left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N} < \infty$, $\int_{B(R)} \left|\frac{\nabla f}{f}\right|^2_{g_N} \mathrm{d}V_{g_N}$ goes to zero as $R \to \infty$. We conclude that f should be a constant. Furthermore, if there exists a point $q \in N$ with $S_N(q) < 0$, then the constant f does not satisfy (3). □

Example. Let (N_1^m, g_1) be a compact manifold with a non-positive scalar curvature. Then $(N, g_N) = (N_1^m \times R^2, g_1 + \delta_{ij})$ satisfies theorem 5 since it has a non-positive scalar curvature and quadratic volume growth.

Remark. The Riemannian–Schwarzschild metric is a Ricci-flat Einstein metric [2, section 9.118]. This metric is given by $N \times_f F$, where $N = R^2$ with the metric $g_N = \mathrm{d}t^2 + 4f'(t)^2\,\mathrm{d}\theta^2$, $(F, g_F) = (S^2, g_0)$ with the standard metric g_0 on S^2, and f satisfies $(f'(t))^2 = 1 - f^{-1}(t)$. The base manifold (R^2, g_N) has positive scalar curvature $S = 2f^3 > 0$. Theorem 5 implies that the sign of scalar curvature of a noncompact base manifold plays an important role in the Einstein warped product construction.

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